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Criticality in elastoplastic models of amorphous solids with stress-dependent yielding rates

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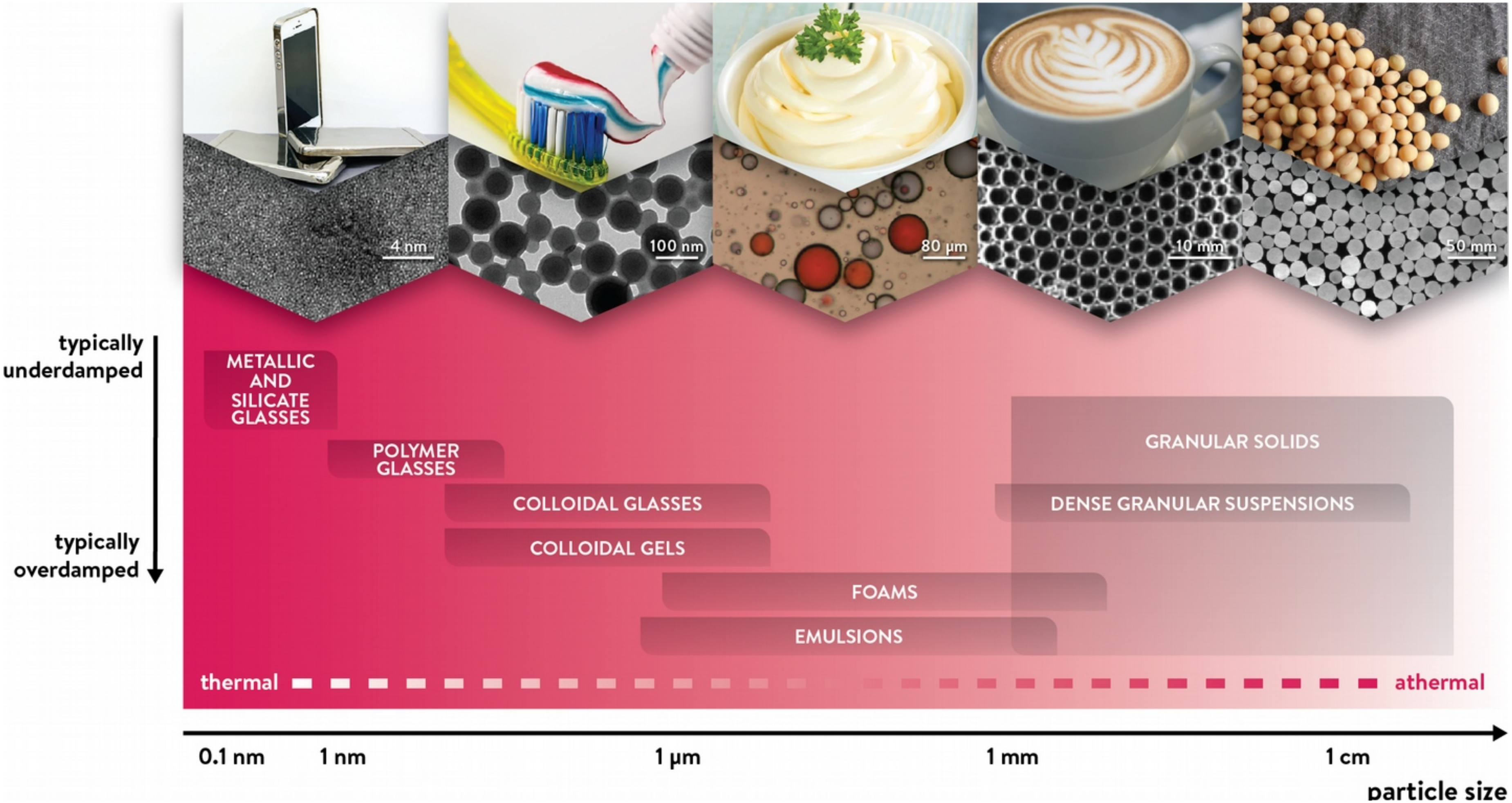
Criticality in elastoplastic models of amorphous solids with stress-dependent yielding rates

E. E. Ferrero *a and E. A. Jagla b

PSM-GM, Grenoble, October 24th 2019

Amorphous materials



very diverse systems... but they share common features

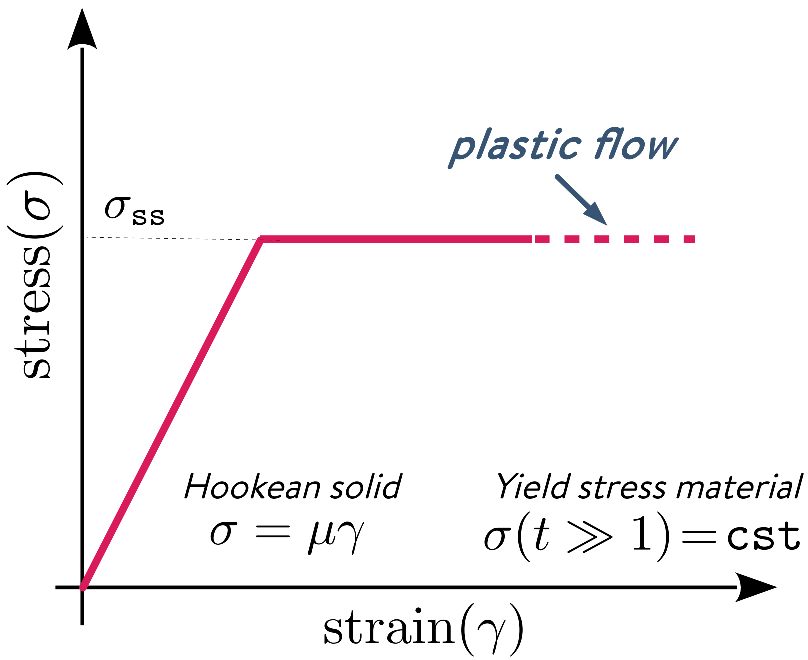
Structurally ***disordered***

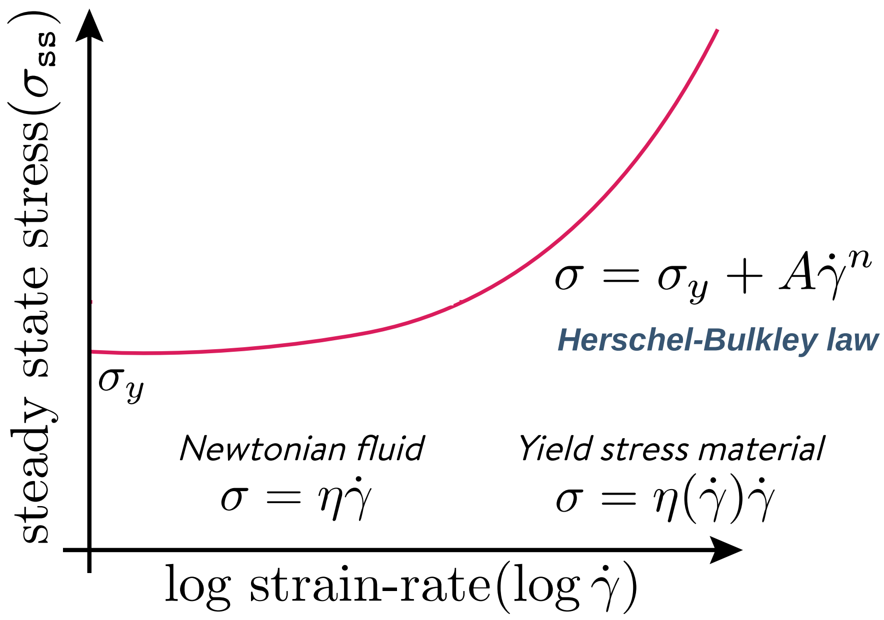
Solid-like (***elastic***) behavior below ***yield stress***

Flow under stress bigger than threshold

A. Nicolas, EEF, K. Martens, J.-L. Barrat, Rev. Mod. Phys. 90, 045006 (2018)

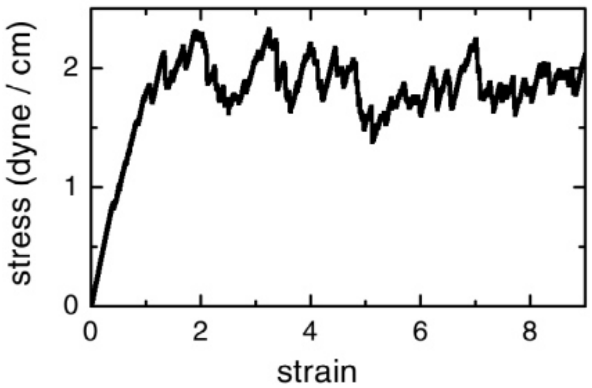
Yield stress systems





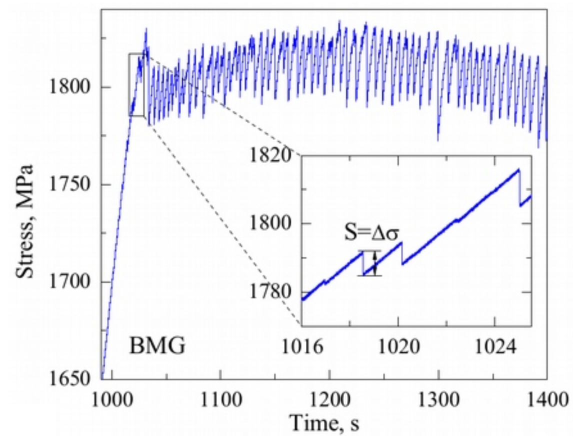
Typical stress-strain and flow curves

Sheared foam



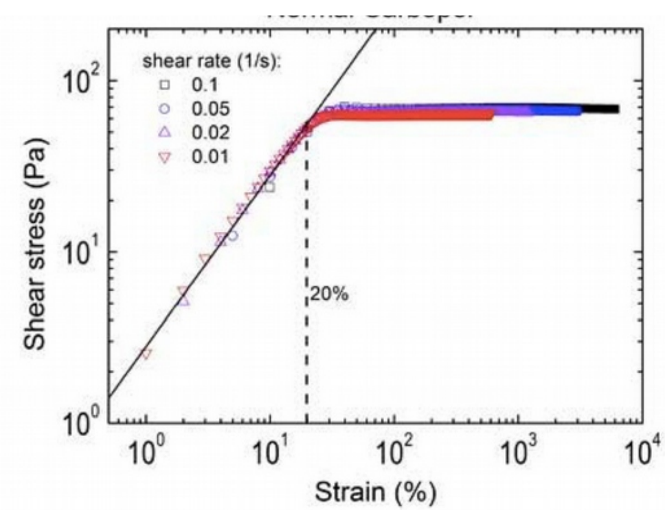
J. Lauridsen et al. PRL **89** 098303 (2002)

Compressed BMG



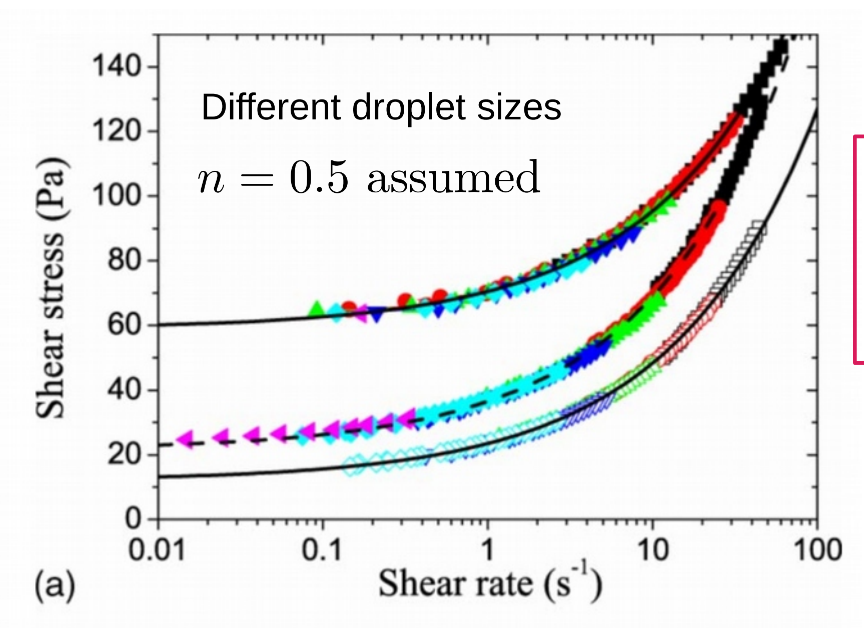
J. Antonaglia et al. PRL **112** 155501 (2014)

Carbopol gel



Dinkgreve et al. Jour. of Rheol. **62**, 773 (2018)

Castor-oil emulsion

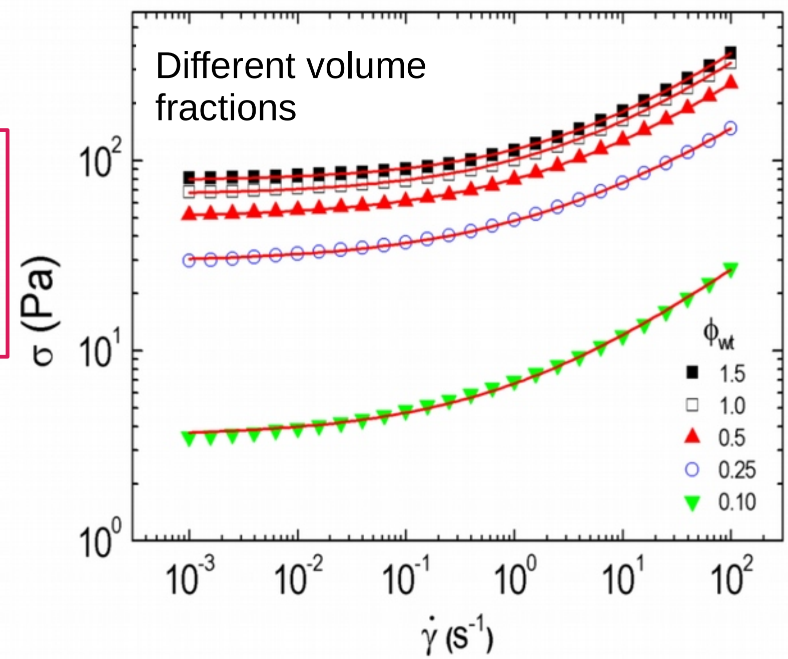


Ovarlez et al. PRE 78, 036307 (2008)

In general:

$$\sigma = \sigma_y + A\dot{\gamma}^n$$

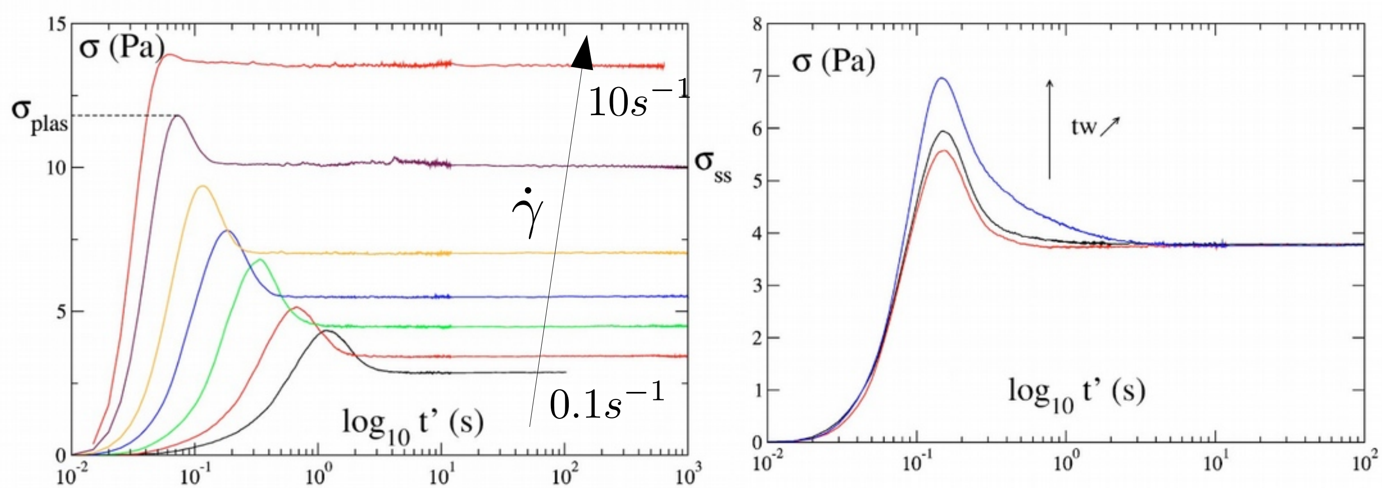
$$n \in [0.4, 0.6]$$



Dinkgreve et al. PRE 92, 012305 (2015)

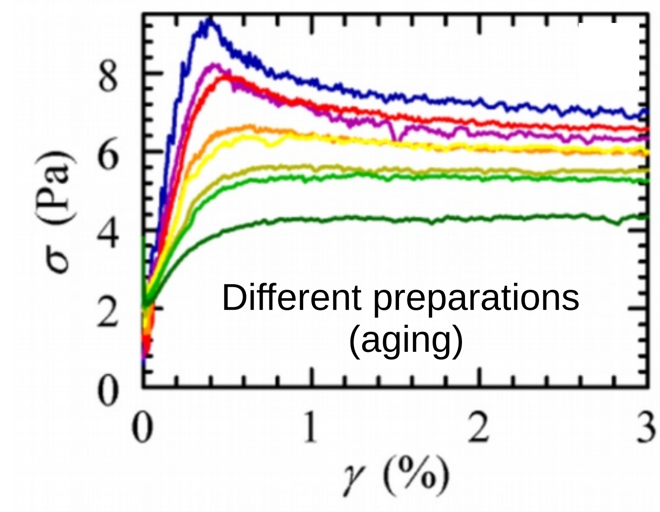
Further complex stress-strain curves

Silica colloids (polymer-stabilized) suspensions



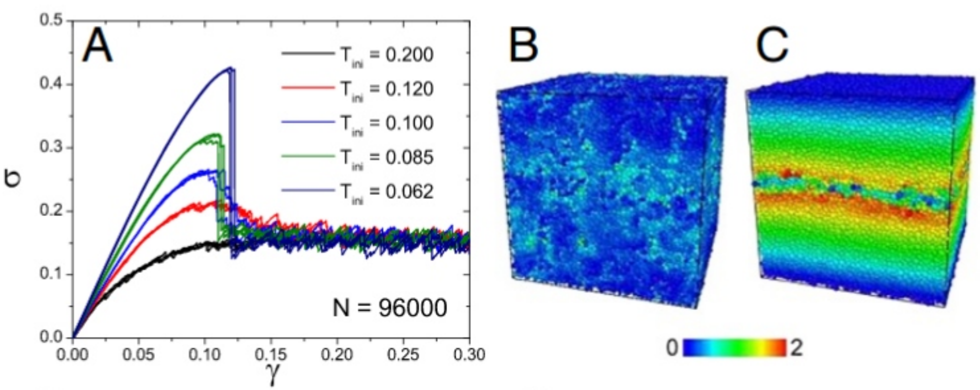
Derec, Ducouret, Adjari, Lequeux PRE 67, 061403 (2003)

Colloidal (carbon black) gel



Sprakel et al PRL 106, 248303 (2011)

Hot Topic: distinction among “brittle” and “ductile” yielding



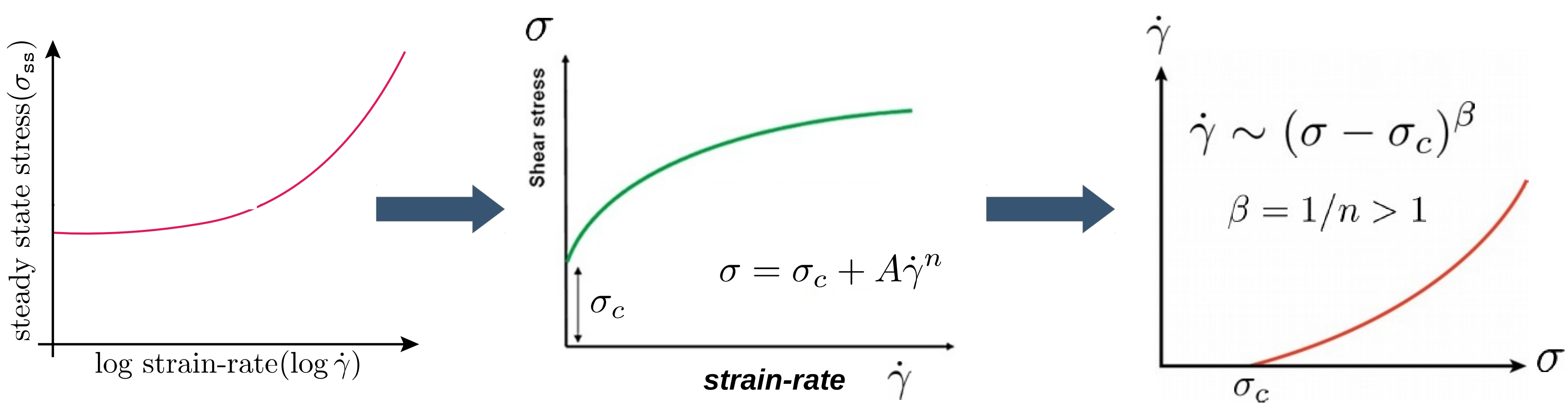
Overshoot, discontinuous stress jump, shear localization, hysteresis, ...

Relevance of the initial configuration

What we do: not-hot topic, the steady state

M. Ozawa, L. Berthier, G. Biroli, A. Rosso, G. Tarjus PNAS **115**, 6656 (2018)

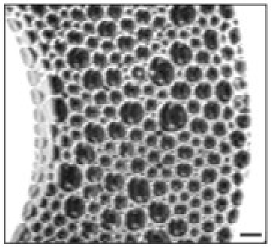
Yielding transition

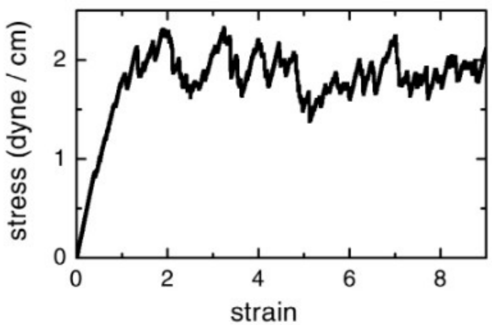


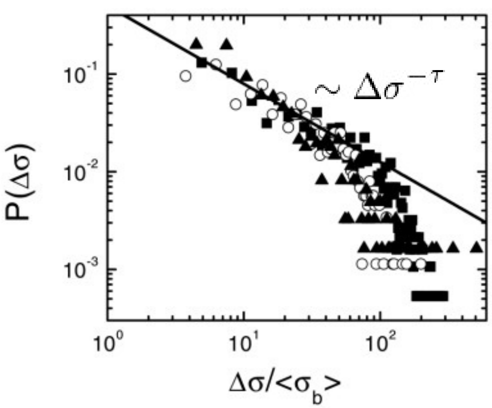
"Yielding transition": a **dynamical phase transition** between an **elastic solid**-like state and a **plastic flow** state when we overcome a **critical yield stress**.

Plastic flow and broadly distributed stress-drop sizes
(avalanches)

Foams

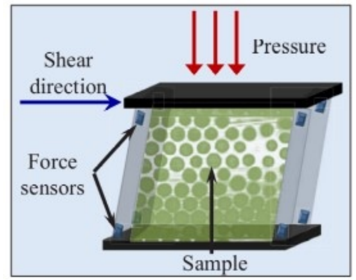


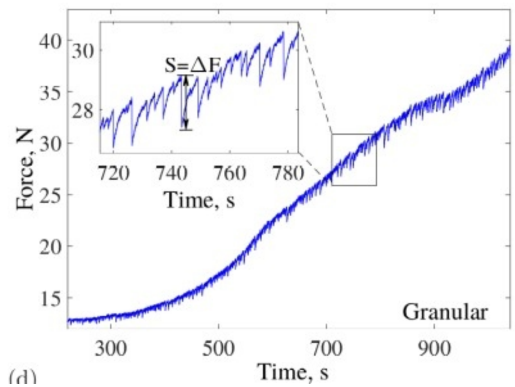


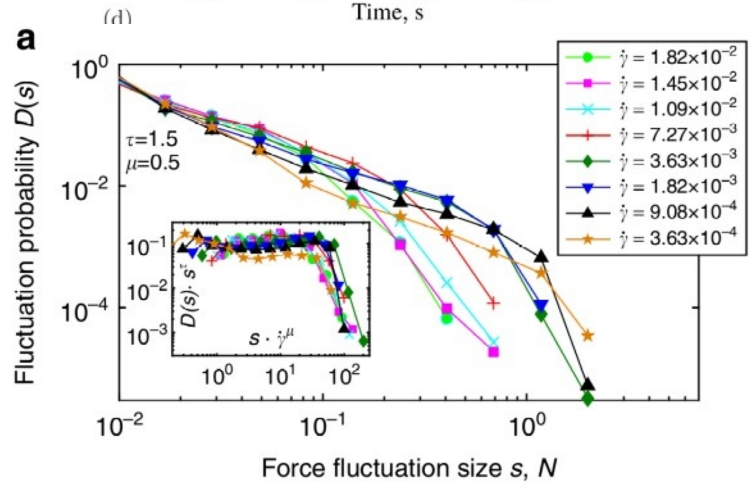


J. Lauridsen et al. PRL **89** 098303 (2002)

Granular systems

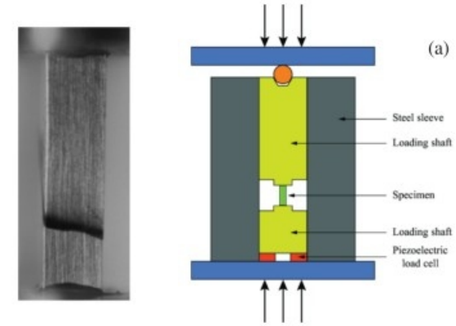


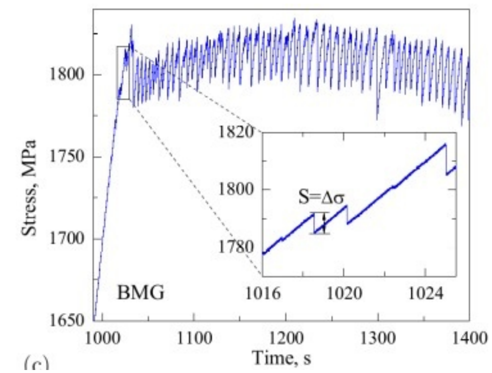


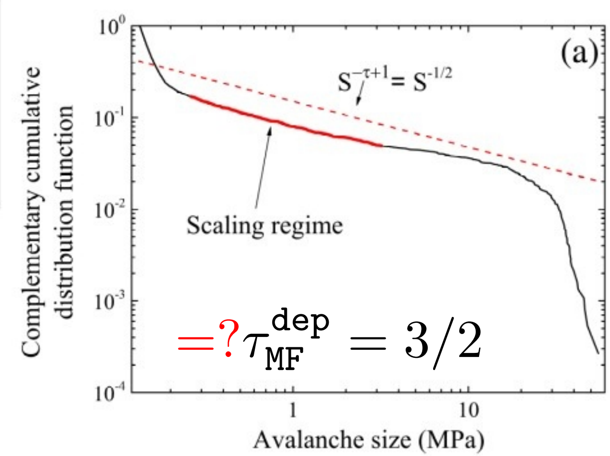


D. Denisov et al. NatComm **7**:10641 (2016), SciRep **7**:43376 (2017)

Bulk metallic glasses

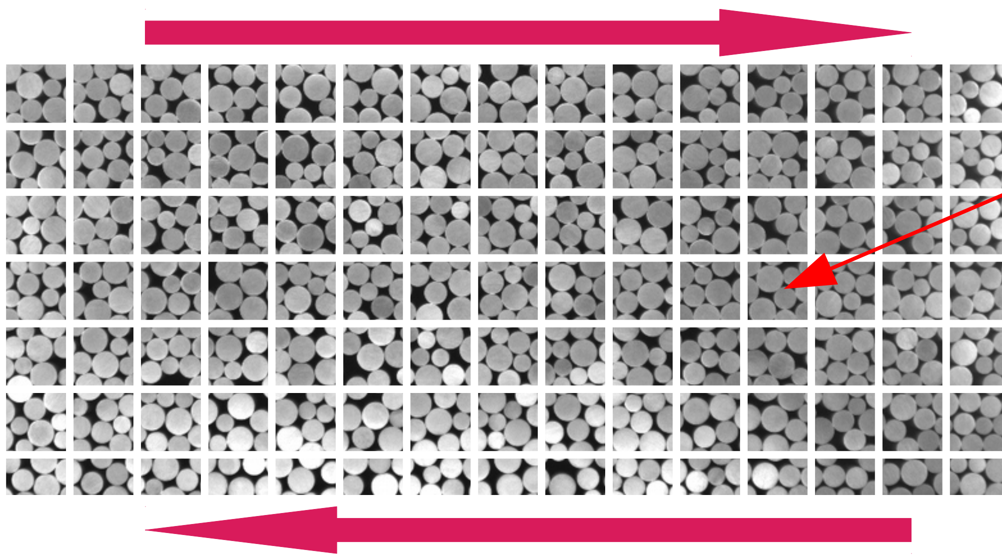






J. Antonaglia et al. PRL **112** 155501 (2014)

A key point: density of shear transformations

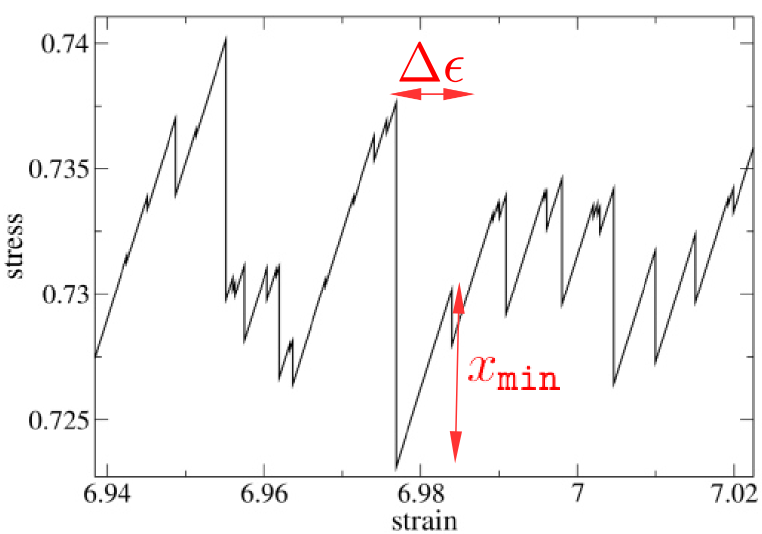


x_i : “distance” to local instability

→ Distribution presents a pseudo-gap

$$\boxed{P(x) \sim x^\theta} \qquad \theta > 0$$

quasistatic



$$\Delta\epsilon \sim x_{\mathtt{min}}$$

- **Observation**:
  The rate at which plasticity occurs is not extensive

$$\langle \Delta\epsilon \rangle \sim \langle x_{\mathtt{min}} \rangle \sim 1/N^\alpha \gg 1/N \ , \quad 0 < \alpha < 1$$

Maloney&Lemaitre *PRL* **93**, 016001 (2004)

- It can be explained by extreme value statistics and pseudo-gap

$$P(x) \sim x^\theta \longrightarrow \langle x_{\mathtt{min}} \rangle \sim N^{-1/(1+\theta)}$$

(the inverse is not necessarily true!)

Karmakar, Lerner, Procaccia PRE **82**, 055103R (2010)

Critical exponents related to yielding

In analogy with equilibrium phenomena and other driven phase transitions

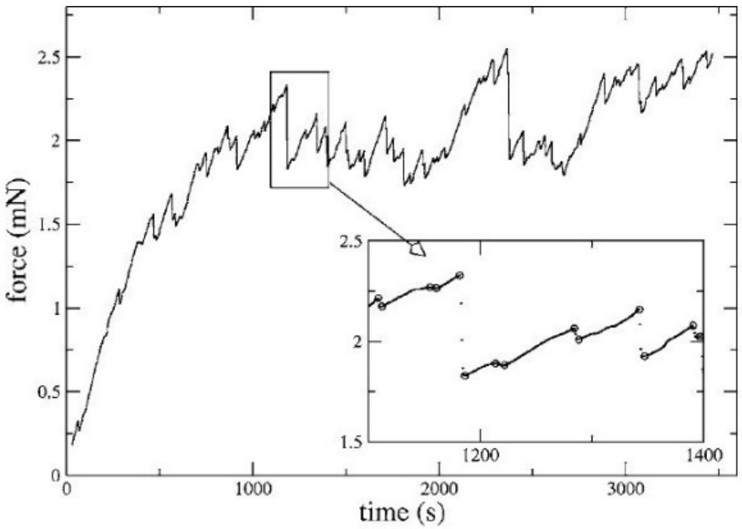
β	$\dot{\gamma} \sim (\sigma - \sigma_c)^\beta$	Flowcurve
θ	$P(x) \sim x^\theta$	Density of shear transformations
$\tau \quad d_f \quad \alpha$	$P(S) \sim S^{-\tau} f(S/S_{\mathtt{cut}})$ $S_{\mathtt{cut}} \sim k^{-\alpha}$ or $S_{\mathtt{cut}} \sim L^{d_f}$	Avalanche size distributions
$\delta \quad z$	$\langle S \rangle \sim \langle T \rangle^\delta \quad \delta = d_f / z$	Avalanche mean size and duration

...and so on

Modeling: basic phenomenology

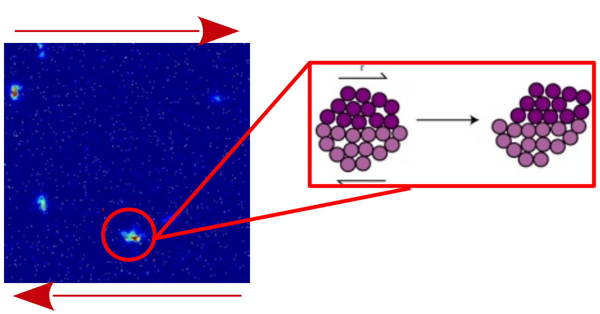
Local rearrangements

➡ "jerky" aspect of the stress response



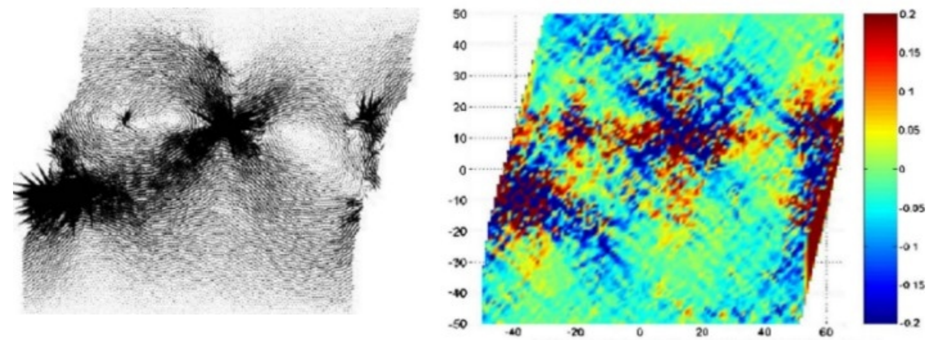
I. Cantat and O. Pitois *Phys. Fluids* **18** 083302 (2006)

➡ well identified, localized **"plastic events",**
a.k.a. shear transformation zones (STZ),
elementary excitations, Eshelbys, ...



A. Nicolas et. al EPJE **37** 50 (2014), Argon and Kuo Mat. Sci. Eng. **39** 101 (1979)

Medium elastic response

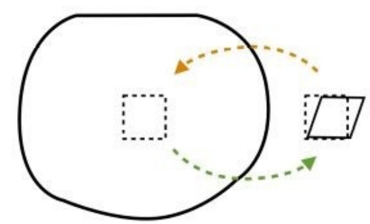


displacement field *stress change*

Maloney and Lemaitre PRL **93**, 195501(2004)

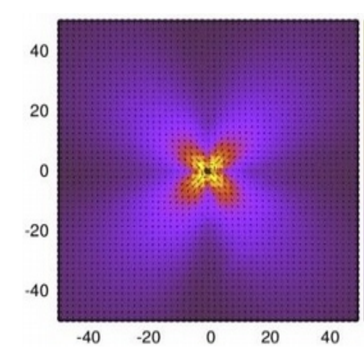
Continuum mechanics:

elastic response to a deformed inclusion



Eshelby propagator for the stress redistribution

$$G^{2D}(r,\theta) \sim \frac{cos(4\theta)}{r^2}$$



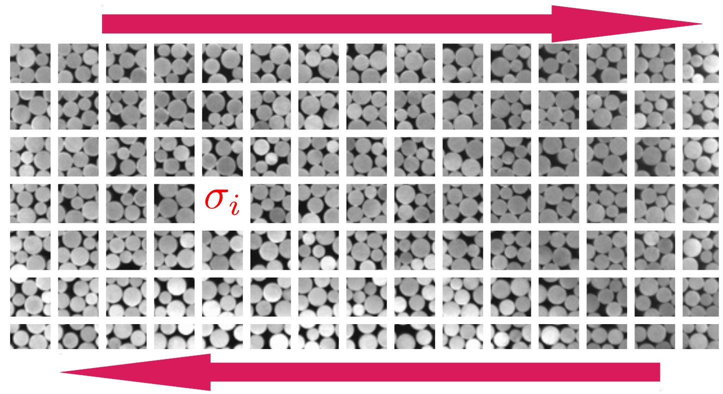
J.D. Eshelby Proc.Roy.Soc. A **241** 376 (1957)

F. Puosi, J. Rottler, J.-L. Barrat PRE **89** 042302 (2014)

Coarse-grained Elasto-Plastic Models (EPM)

Simplifications:

- *Meso-scale*
- *Scalar*
- *Athermal*
- *Overdamped*
- *...*



- configuration $\{\sigma_i\}$
- elastic loading
- local yielding (plastic event) and stress redistribution

$$\partial_t \sigma_i(t) = \mu \dot{\gamma}^{\mathrm{ext}} - g_0 n_i(t) \frac{\sigma_i(t)}{\tau} + \sum_{j \neq i} G(i,j) n_j(t) \frac{\sigma_j(t)}{\tau}$$

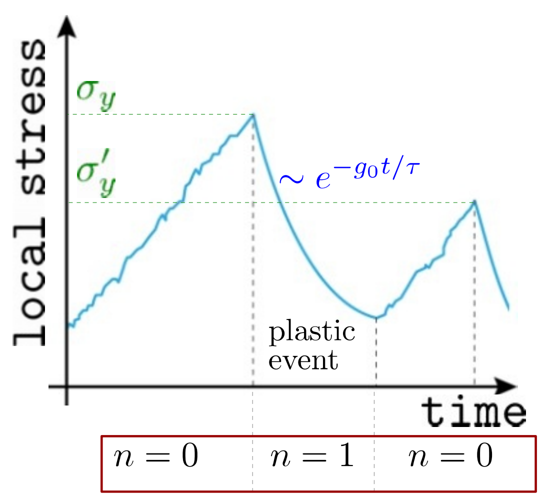
external strain-rate — local plastic yield — "**mechanical noise**" due to plastic activity elsewhere

+ Dynamical **rules** for a "local state" n_i

$$n_i : \begin{cases} 0 \to 1 \text{ typically when } \sigma_i > \sigma_{y_i} \\ 1 \to 0 \text{ e.g., after a time } \tau_1 \end{cases}$$

$n_i(t) = 0$ locally elastic

$n_i(t) = 1$ locally plastic

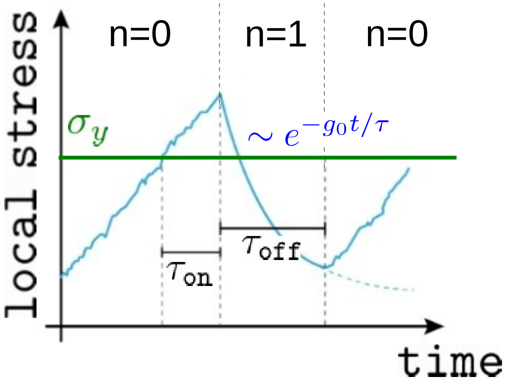


$$G_{ij}^{2D} = \frac{\cos(4\theta_{ij})}{\pi r_{ij}^2}$$

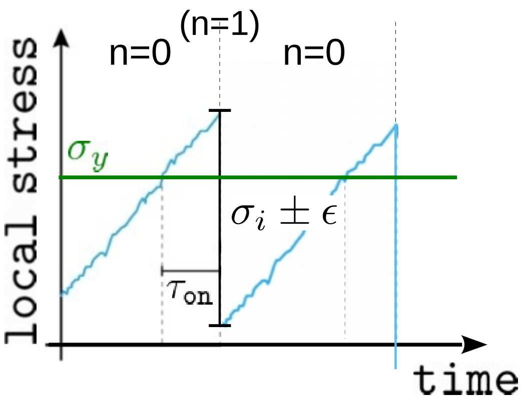
Eshelby **propagator**

EPM with stress-dependent rates

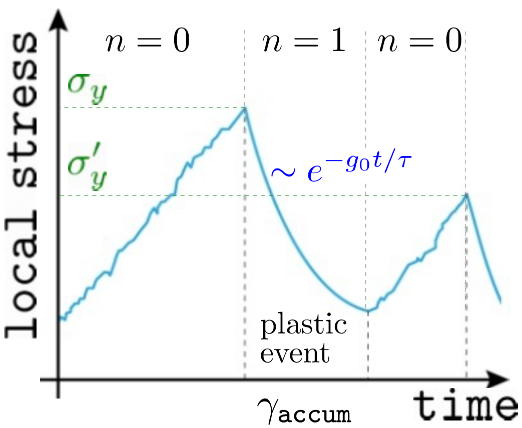
Picard's model



Lin's model



Nicolas' model



EEF & EA Jagla Soft Matter (2019)

$$\partial_t \sigma_i(t) = \mu \dot{\gamma}^{\text{ext}} - g_0 n_i(t) \frac{\sigma_i(t)}{\tau} + \sum_{j \neq i} G(i,j) n_j(t) \frac{\sigma_j(t)}{\tau}$$

Stochastic rules for local yielding:

$$g_0 \equiv -G_{ii} > 0$$

Uniform rate

$$\tau_{\text{on}}^{-1} = \lambda = \text{cst.}$$

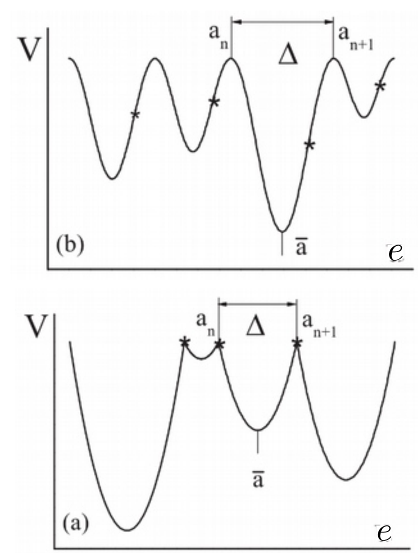
(all previous cases)

Progressive rate

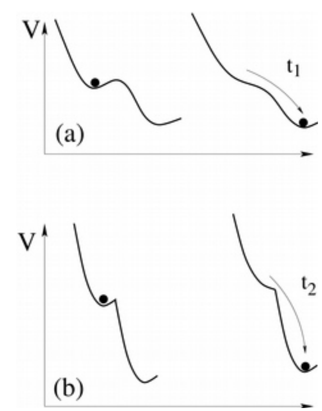
$$\tau_{\text{on}}^{-1} = \lambda(\sigma) \propto \sqrt{\sigma - \sigma_y}$$

New: site is more likely to yield as it is more overloaded

Similar to a sliding manifold on ***parabolic*** or ***smooth*** potentials



$$\eta \dot{e}_i = -\frac{dV_i(e_i)}{de_i} + \sum_j G_{ij} e_j + \sigma,$$



$$t_1 \sim \delta\sigma^{-1/2}$$

$$t_2 \sim \text{cst.}$$

Jagla, JSTAT 013401 (2018), Fernández Aguirre and Jagla, PRE 98, 013002 (2018)

Simulation protocols

EEF & EA Jagla Soft Matter (2019)

We simulate 6 different EPMs:
Picard's, Lin's and Nicolas' on their classic version and with progressive rates

Finite strain-rate

$$\frac{\partial \sigma_i(t)}{\partial t} = \mu \dot{\gamma}^{\text{ext}} + \sum_j G_{ij} n_j(t) \frac{\sigma_j(t)}{\tau};$$

Solved with a pseudo-spectral method:
back and forth from Fourier space

$$G_{\mathbf{q}} = -\frac{4 q_x^2 q_y^2}{(q_x^2 + q_y^2)^2}. \qquad G_{\mathbf{q}=0} = -\kappa$$

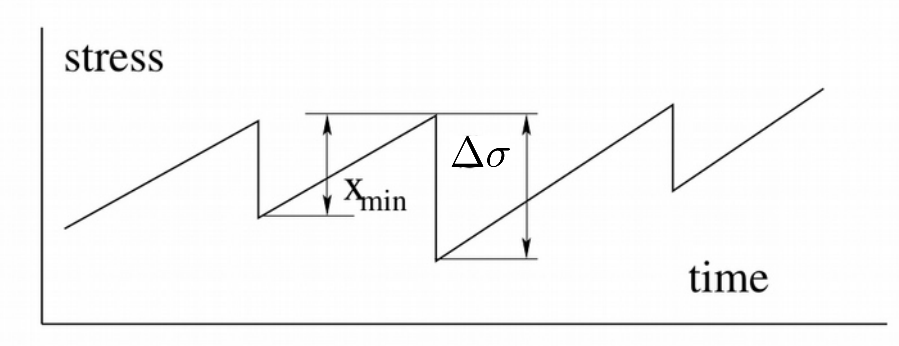
κ: stress non-conservation parameter

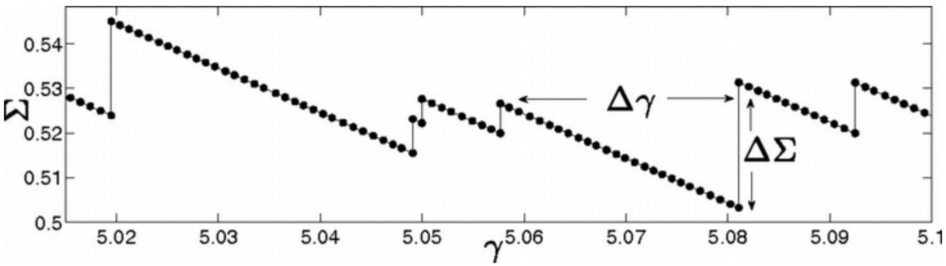
typically $\kappa = 1$

Used for: flowcurves

Quasistatic

Ideally: $\dot{\gamma} = 0$

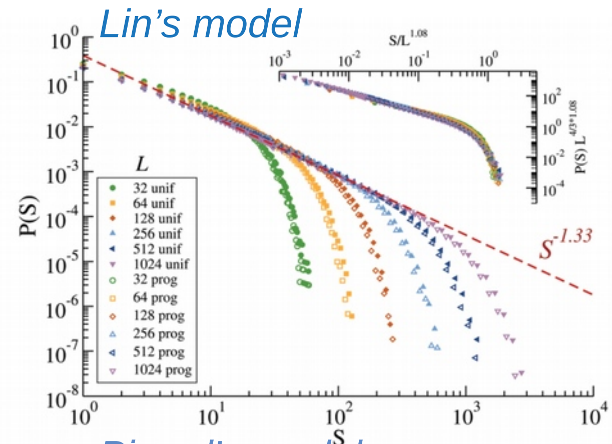


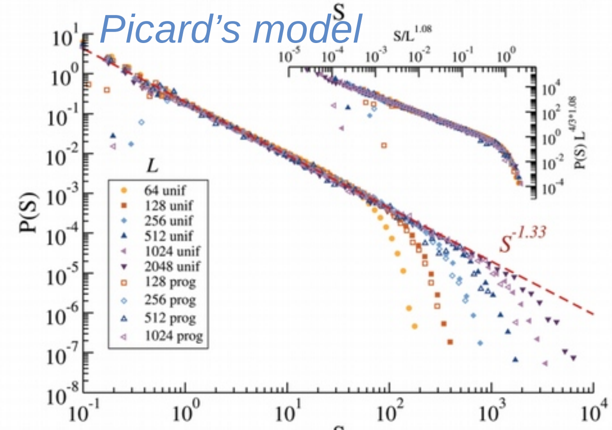


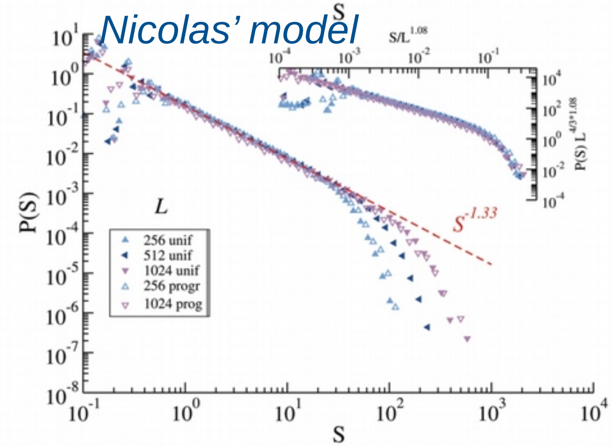
[Lin et al. PNAS 111 14382 (2014)]

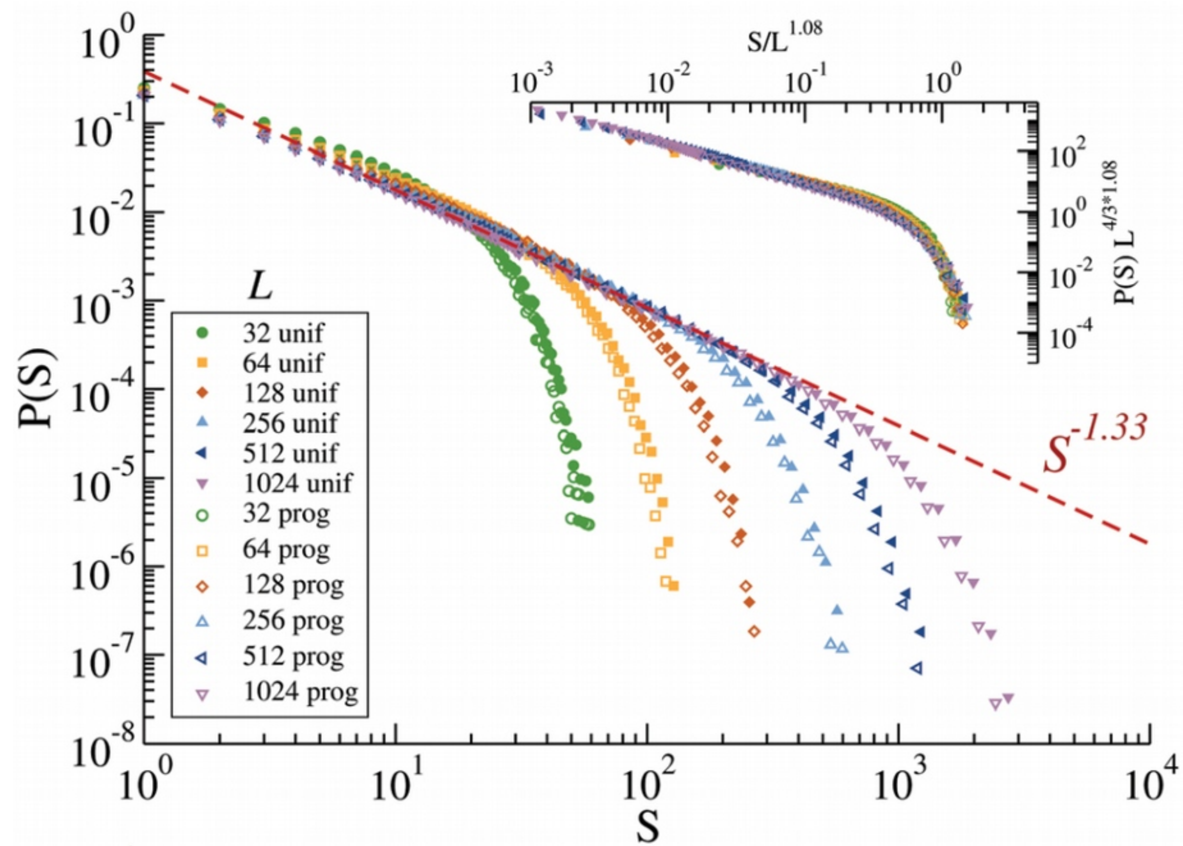
Used for: avalanche statistics

Quasistatic avalanches size distribution









$$P(S) \sim S^{-\tau} f(S/S_{\mathrm{cut}}) \qquad S \equiv \Delta\sigma L^d = \Delta\sigma N$$

$$S_{\mathrm{cut}} \sim L^{d_f}$$

$$\to \tau \simeq 1.33$$

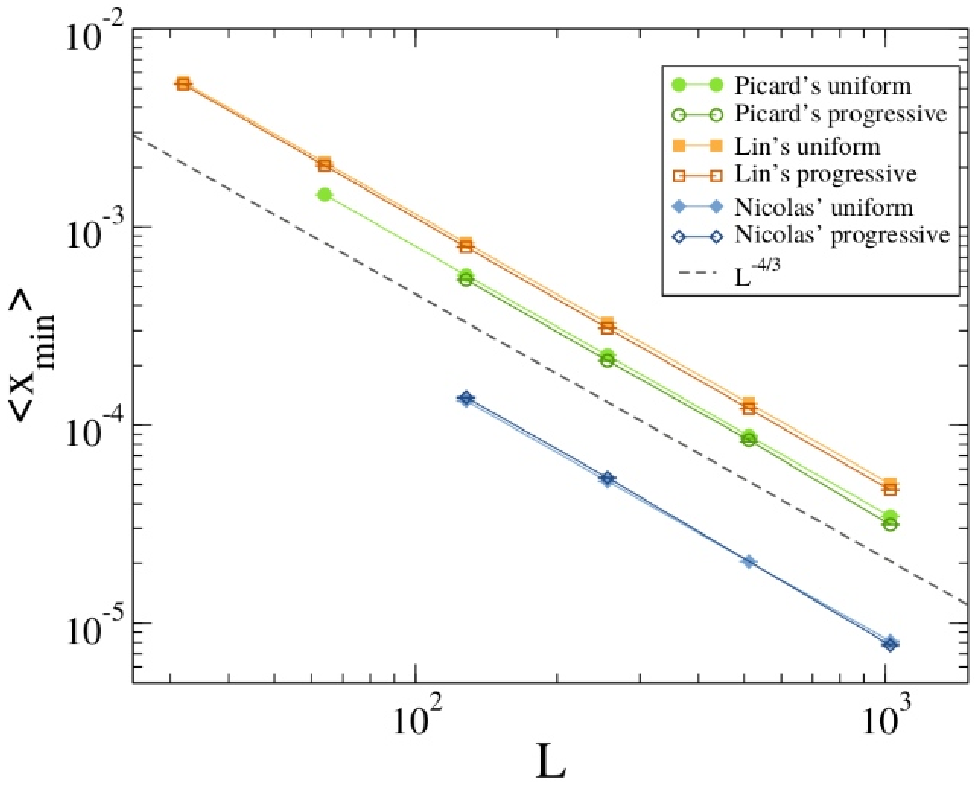
$$\to d_f \simeq 1.08$$

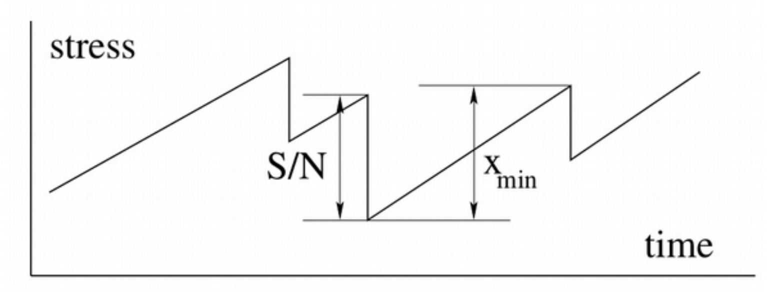
- **Independent** on rate rule, for all 3 models

- τ and d_f can be **sensitive** to fit criteria and finite size effects, but are **universal**

movie

Extremal statistics of distances to threshold





stationarity → $\langle x_{\mathtt{min}} \rangle = \langle \Delta\sigma \rangle$

$$\langle x_{\mathtt{min}} \rangle \sim N^{-1/(1+\vartheta)} \sim L^{-d/(1+\vartheta)}$$

$$\langle \Delta\sigma \rangle \sim \langle S \rangle / N \sim L^{d_f(2-\tau)-d}$$

$$\langle x_{\mathtt{min}} \rangle \sim N^{-\phi} \sim L^{-d/(1+\vartheta)}$$

$$L^{-4/3} \quad \rightarrow \vartheta \simeq 0.5$$

$$\rightarrow \tau \simeq 1.33$$

$$\rightarrow d_f \simeq 1.08$$

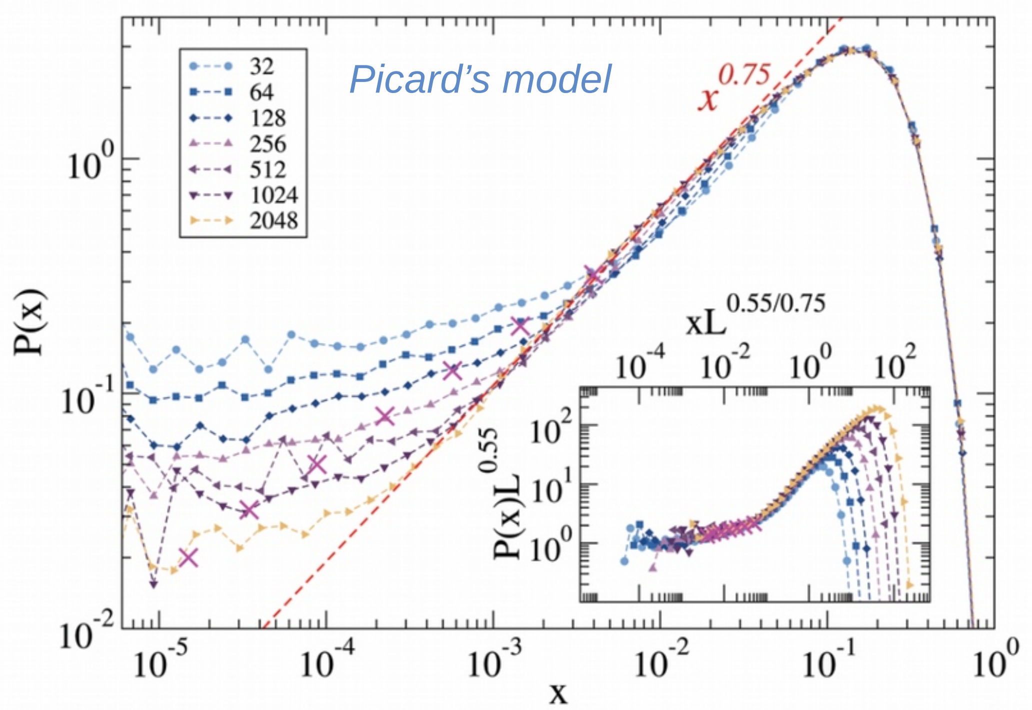
$$\tau = 2 - \frac{\vartheta}{\vartheta + 1} \frac{d}{d_f}$$

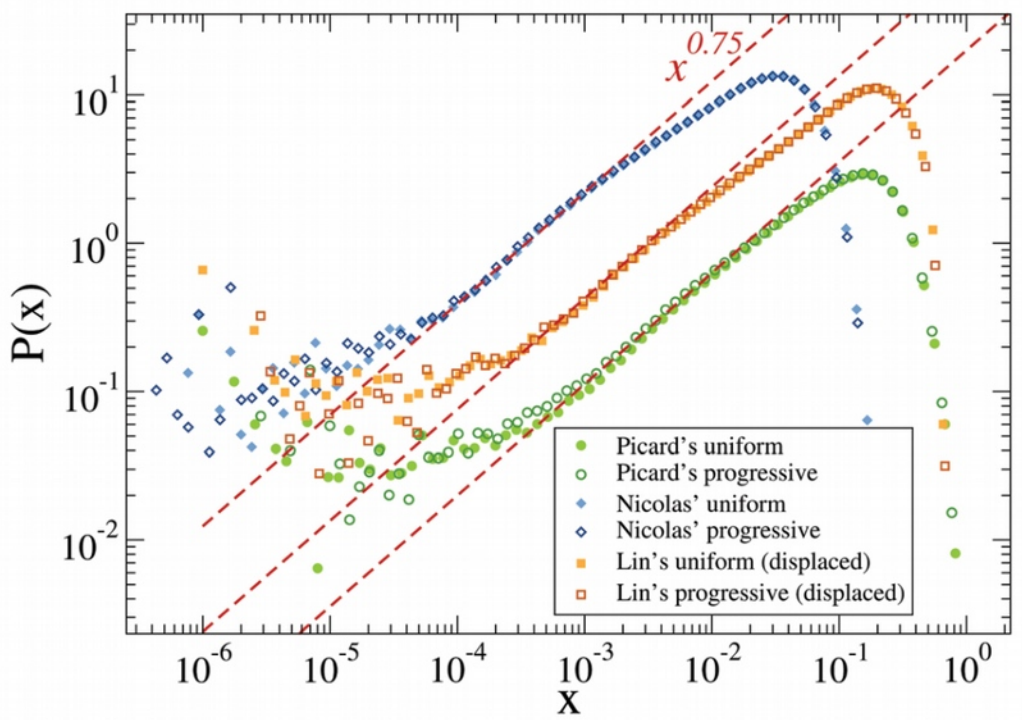
holds (within errors)

- **Independent** on rate rule, for all 3 models
- Independent on the model: **universal**

τ, d_f, ϑ 'static' critical exponents

Density of shear transformations P(x)





$$P(x) \simeq P(0) + x^{\theta}$$

$$P(0) \longrightarrow_{N\to\infty} 0 \qquad P(0) \sim N^{-a}$$

The $<x_{min}>$ values (crosses) are largely determined by the plateau scaling

A proposed θ=*0.75* finds compatibility with all models in the range of *x* previous to the plateau

- **Independent** on rate rule, for all 3 models
- (Possibly) independent on the model.

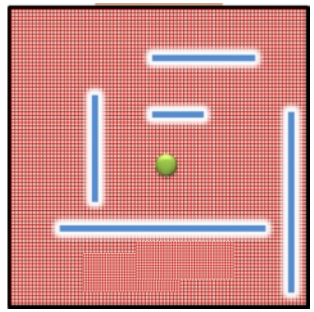
Also: Tyukodi et al. ArXiv:1905.07388, Ruscher&Rottler 1908.01081

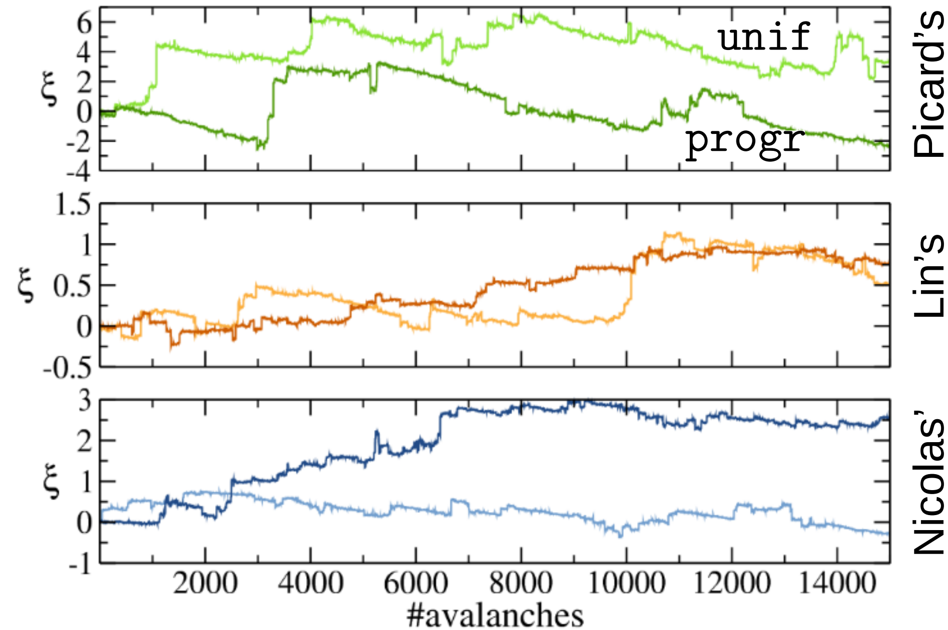
Accumulated noise and mean-field description

Accumulated noise at a given point

$$\xi_i = \sum_{\text{time}} \delta\xi_i$$

"time" = avalanche index





Let's describe it as a correlated noise with "Hurst" exponent *H* (random walk has $H=0.5$)

$$\xi(kx) = k^H \xi(x)$$

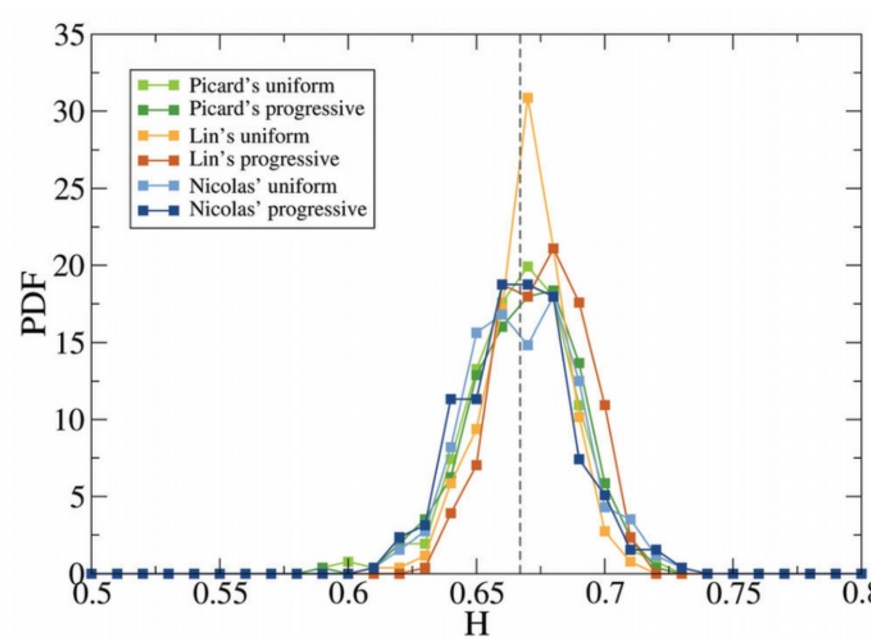
In a mean field description, $H=1/\mu$ where each "kick" contributing to ξ comes from a distribution

$$P(\delta\xi) \sim \frac{1}{|\delta\xi|^{\mu+1}} \qquad \mu = \frac{1}{H}$$

[Lin&Wyart, PRE **97**, 012603 (2018)]

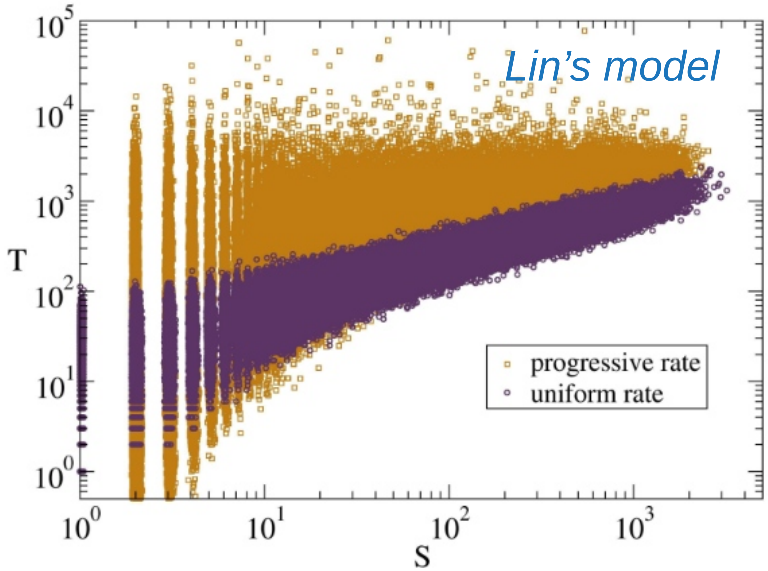
If we accept that those "kicks" come from far avalanches (instead of single-site events), we find a consistent MF description

$$\partial_t \sigma(t) = \mu \dot{\gamma}^{\text{ext}} - g_0 n(t) \frac{\sigma(t)}{\tau} + \delta\xi$$



$H \simeq 0.67$ is found for all models and variants

Avalanche durations (‘dynamical’ exponent z)



$$S \sim \ell^{d_f} \qquad T \sim \ell^{z} \qquad T \sim S^{z/d_f}$$

Uniform rate

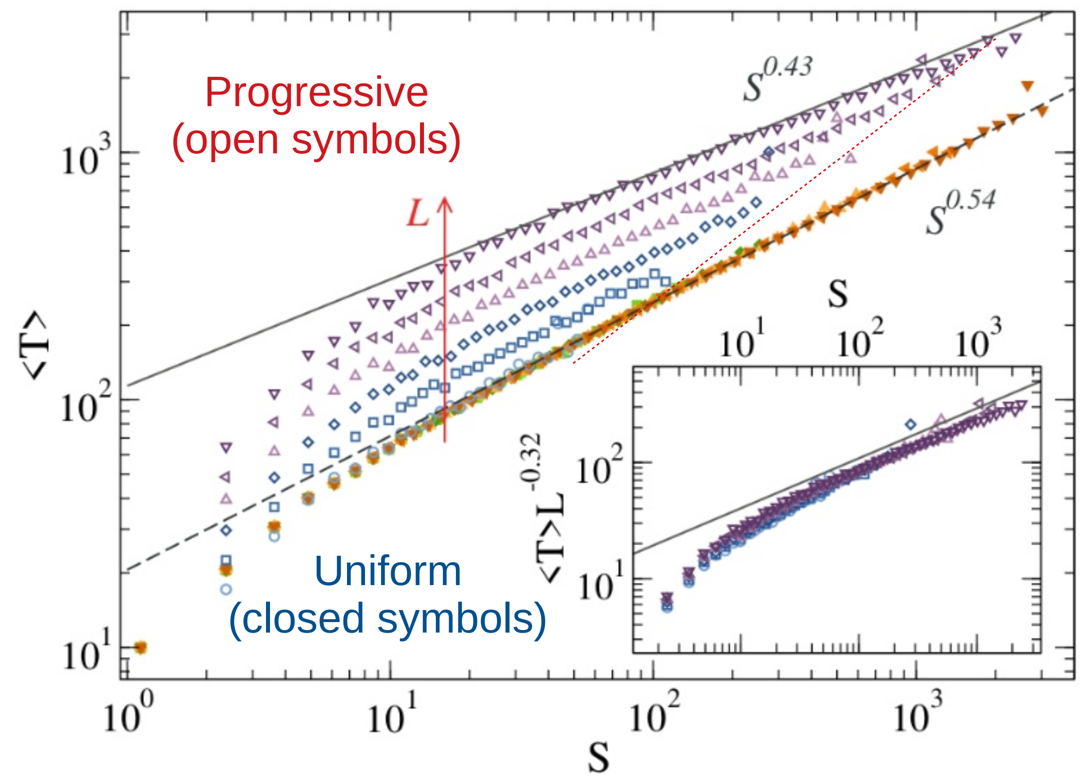
$$z/d_f \simeq 0.54 \quad z \simeq 0.58$$

Progressive rate

$$z/d_f \simeq 0.43 \quad z \simeq 0.46$$

$$z'/d_f \simeq 0.7 \quad z' \simeq 0.76$$

z depends on the yielding rule
Exponents differ, but also the behavior with *L*



New event in the avalanche, time *dT* added

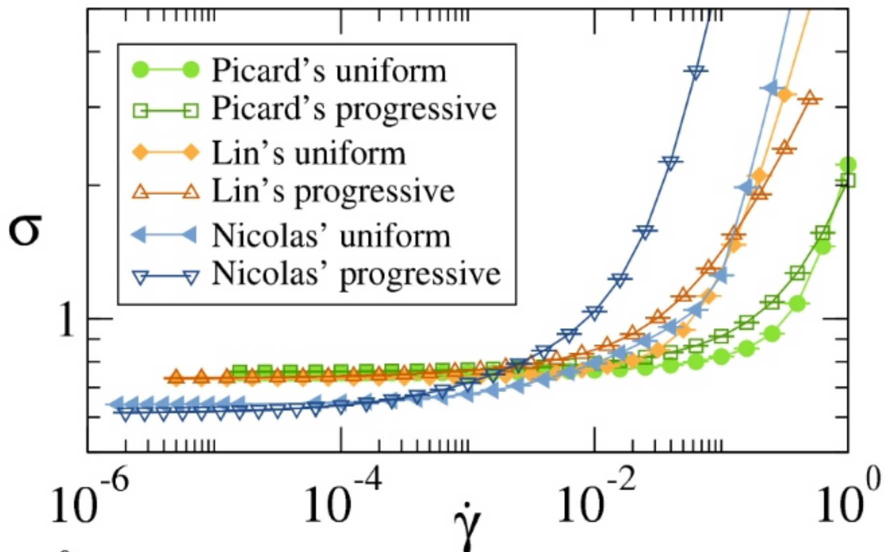
Uniform *Progressive*

$$dT \simeq \lambda^{-1} \sim 1 \qquad dT \simeq \lambda^{-1} \sim (\sigma - \sigma_Y)^{-1/2}$$

For progressive rates events that most contribute to the total duration have a small probability of happening but their observation increases with system size

$$T \sim N^{\alpha} S^{(1-\alpha)/2}$$

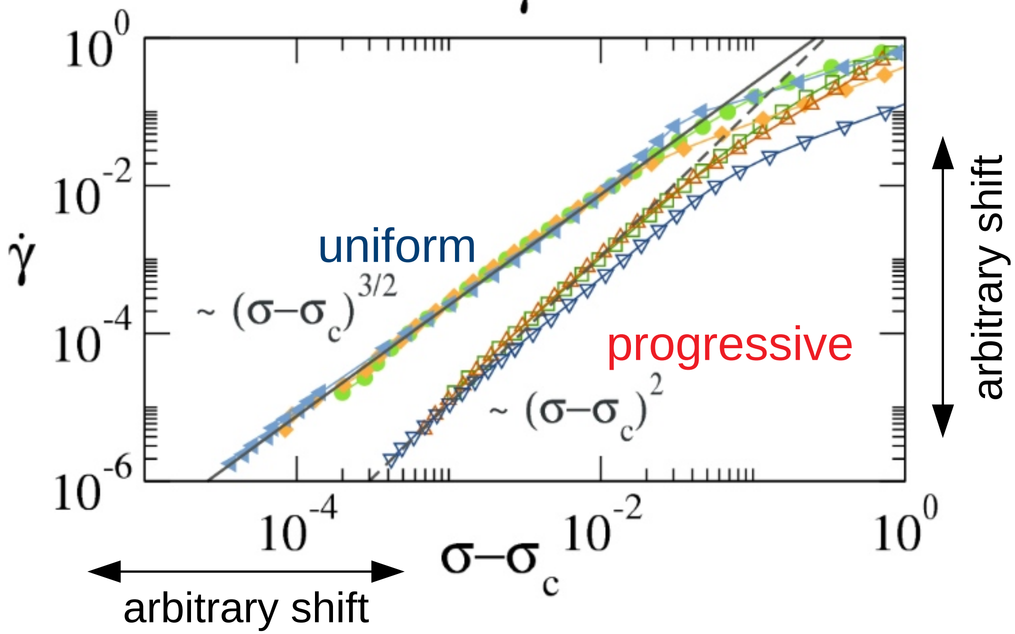
Flowcurves (β exponent)



$$\langle\sigma(\dot\gamma)\rangle \propto \sigma_c + A\dot\gamma^n \qquad \dot\gamma \propto (\sigma-\sigma_c)^\beta \qquad \beta = 1/n$$

Uniform rate	*Progressive rate*
$\lambda = \texttt{cst.}$	$\lambda(\sigma) \propto (\sigma-\sigma_y)^{\frac{1}{2}}$
$\beta = 1.5$	$\beta = 2$

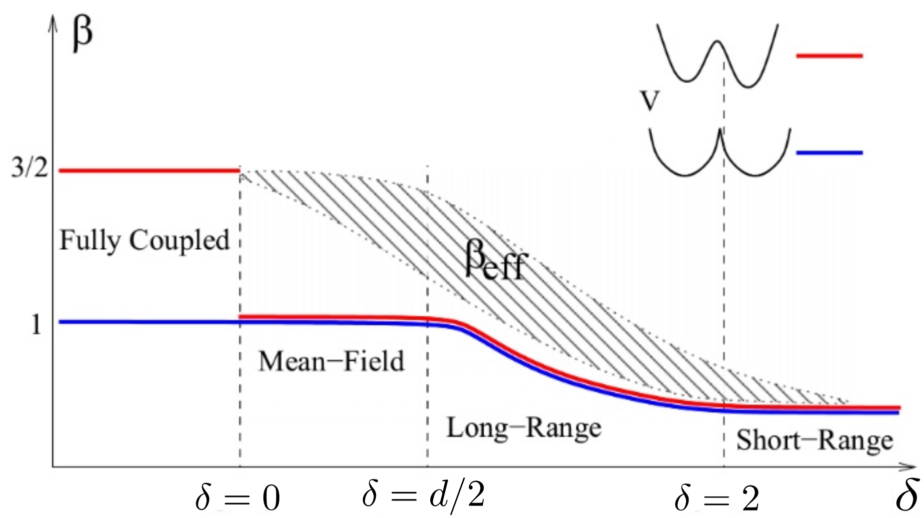
interestingly more similar to measured values *n~0.5*



β depends on the yielding rule

In Fully Coupled - MF depinning also, two different flow exponents *β* coexist

$$G_{ij} = \kappa/|i-j|^{d+\delta}$$



A.B. Kolton and E.A. Jagla, PRE 98, 042111 (2018)
D.S. Fisher, Phys. Rev.p 301, 113 (1998)

Relation among exponents

We have already derived

$$\tau = 2 - \frac{\vartheta}{\vartheta+1}\frac{d}{d_f}$$

A fractional Brownian motion with Hurst exponent *H=1/μ,* signed jumps

$$P(\delta\xi) \sim \frac{1}{|\delta\xi|^{\mu+1}}$$

and an absorbing boundary at *x=0* generates a *P(x)~x^θ* with

$$\theta = \mu/2$$

Also, in mean-field* $\beta = \mu$ for $1 < \mu < 2$

In general, taking into account the type of rate rule**

$$\beta = \frac{1}{H} + 1 - \frac{1}{\omega}$$

ω=1 for uniform rates, *ω=2* for progressive

For the HL model it can be shown that changing

$$\nu_{\mathrm{HL}}(\sigma,\sigma_c) \equiv \frac{1}{\tau}\Theta(\sigma-\sigma_c) \quad \text{to} \quad \nu(\sigma,\sigma_c) \equiv \frac{1}{\tau}(\sigma-\sigma_c)^{\eta}\Theta(\sigma-\sigma_c)$$

we obtain $\dot{\gamma} \sim (\sigma - \sigma_c)^{\eta+2}$

Rate type	PT ($H = 1$)	HL ($H = 1/2$)	2d-EPM ($H = 2/3$)
Uniform	1	2	3/2
Progressive	3/2	5/2	2

β exponent. Prandtl-Tomlinson (PT), Hébraud-Lequeux (HL)

Exponents for 2*d* EPMs

	Uniform	Progressive
β	1.5 ± 0.1	2.0 ± 0.1
ϑ (from $\langle x_{\min}\rangle$)	0.5 ± 0.05	0.5 ± 0.05
θ (from $P(x)$)	0.75 ± 0.07	0.75 ± 0.07
τ_S	1.33 ± 0.03	1.33 ± 0.03
d_f	1.08 ± 0.05	1.08 ± 0.05
H	0.67 ± 0.03	0.67 ± 0.03
z/d_f	0.54 ± 0.02	0.43 ± 0.04
z	~ 0.583	~ 0.464

What's the relation between ϑ and θ or H?

$$H = 1/(3-\tau)?$$

*Lin&Wyart, PRE **97**, 012603 (2018)
**Jagla JSTAT 013401 (2018), Fernández&Jagla PRE 98, 013002 (2018)

Summary

- **Amorphous solids** undergo a **yielding transition** characterized by a power-law vanishing of the flowcurve at a finite stress**.**

- **Some consensus** has been reached in avalanche statistics exponents for **yielding** in EPMs (and MD) simulations in the quasistatic limit; **still unclear in experiments**.

- Within EPMs, different **local yield rules** strongly affect "dynamical" exponents (β, z) but not the static critical ones (τ, d_f, ϑ, θ, H)

- *P(x)* suffers from strong finite-size effects and the picture is **not as simple** as $\sim x^{\theta}$.

- **Yielding in finite dimensions** can be described as an **effective (not-depinning) mean-field**, were we plug-in a dimension-dependent Hurst exponent to mimic the avalanche-induced noise.

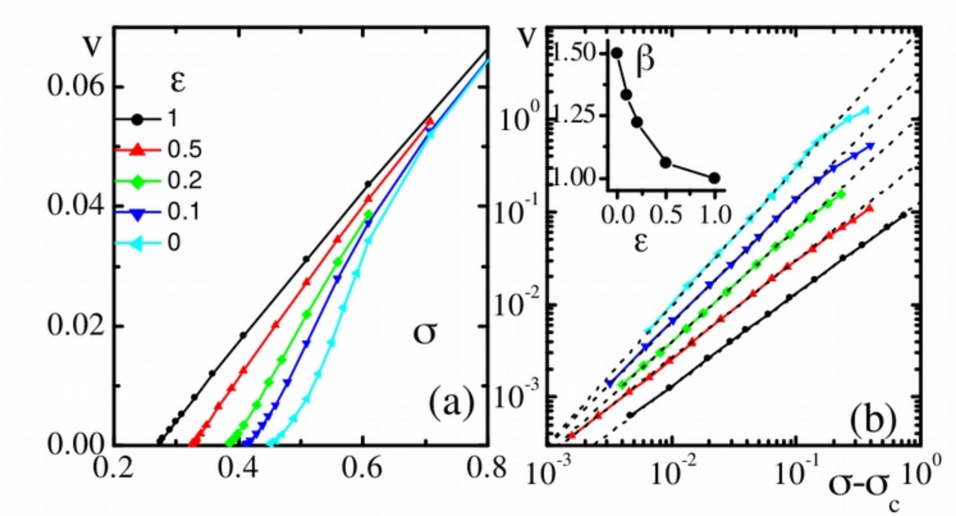
Elastic interfaces on disordered substrates: From mean-field depinning to yielding

EE Ferrero and EA Jagla, arXiv:1905.08771*

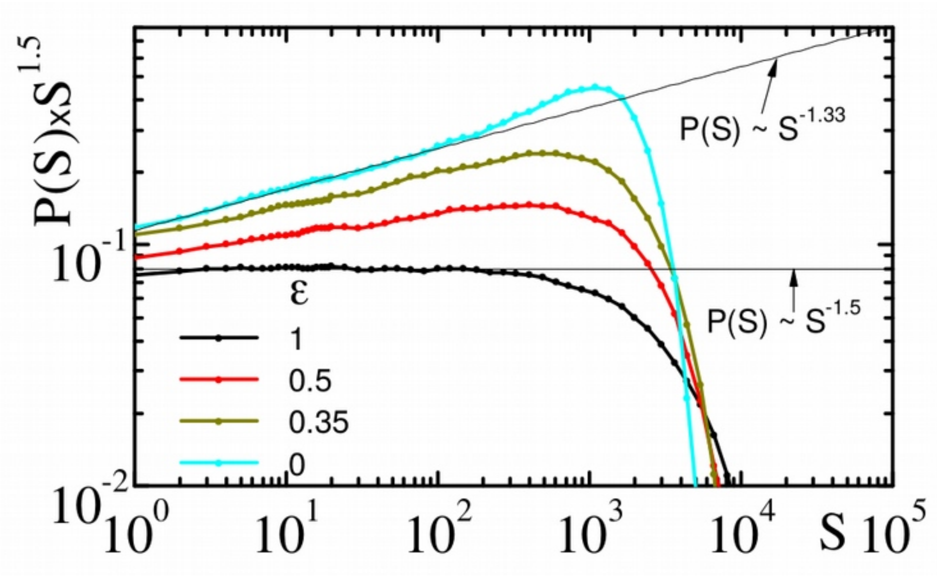
*not yet rejected from PRL

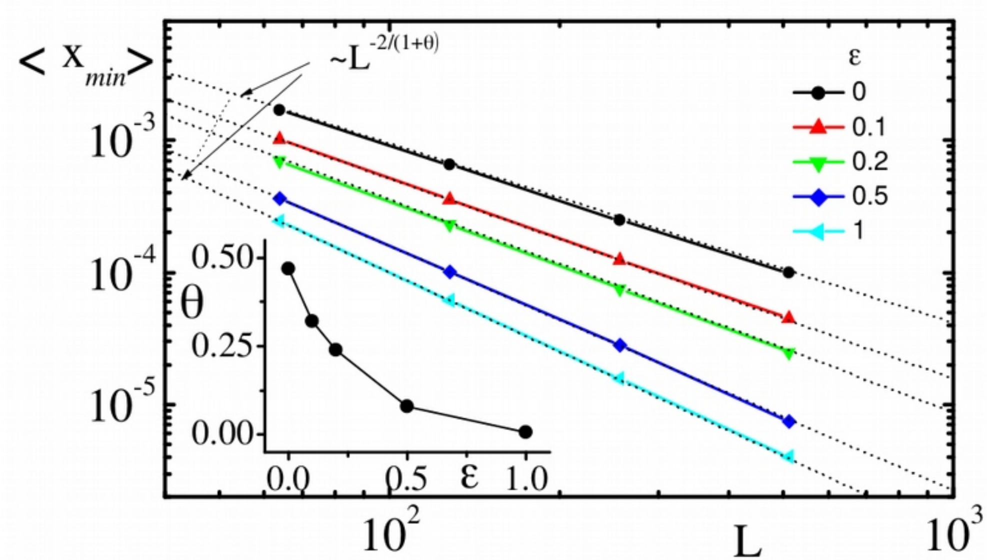
$$G_{\mathbf{q}} \equiv (1-\varepsilon)G^{\mathrm{Y}}_{\mathbf{q}} + \varepsilon G^{\mathrm{MFD}}_{\mathbf{q}}$$

$$G^{Y}_{\mathbf{q}} = -\frac{(q_x^2 - q_y^2)^2}{(q_x^2 + q_y^2)^2} \qquad G^{\mathrm{MFD}}_{\mathbf{q}} = -1$$



...and also depend on the kind of potential: cuspy (uniform rates) or soft (progressive r.)





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